Additional Practice CS103 Final II

This practice exam is not worth any extra credit points, but should be good practice for the final exam. This was the exam that was given in Fall 2012.

Problem One: Discrete Mathematics (15 Points)

Prove, by induction on n, that for any $n \in \mathbb{N}$ with $n \geq 2$, that

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_n (n+1) = \log_2 (n+1)$$

You might want to use the change of base formula for logarithms: for any $c \in \mathbb{R}$ with $c > 1$,

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Problem Two: Regular Languages (40 Points Total)

Suppose that you really, *really* don't like the string **abba** and want to build a language of everything except that string. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider the language $NOT_{\mathtt{abba}}$ defined as follows:

$$NOT_{\mathtt{abba}} = \{ w \in \Sigma^* \mid w \neq \mathtt{abba} \}$$

For example, $\varepsilon \in NOT_{\mathtt{abba}}$ and $\mathtt{abbabb} \in NOT_{\mathtt{abba}}$, but (unsurprisingly) $\mathtt{abba} \notin NOT_{\mathtt{abba}}$.

(i) Finite Automata (10 Points)

Design a DFA for the language $NOT_{\mathtt{abba}}$.

(ii) Regular Expressions (10 Points)

Write a regular expression for $NOT_{\mathtt{abba}}$.

(iii) Nonregular Languages (20 Points)

Let $L = \{\mathtt{0}^n\mathtt{1}^n \mid n \in \mathbb{N}\}$. Prove that no infinite subset of L is a regular language. This result shows that not only is it impossible to write a regular expression for L, but it is also impossible to write a regular expression that matches infinitely many strings from L without also matching at least one string not in L.

Problem Three: Context-Free Languages (25 Points Total)

(i) Context-Free Grammars (15 Points)

Consider the following language L over the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$:

$$L = \{\ w \in \Sigma^* \ \mid\ |w| \equiv_4 0\text{, and the first quarter of the characters in } w \text{ contains at least one } \texttt{b}\ \}$$

For example, $\underline{\texttt{b}}\texttt{aaa} \in L$, $\underline{\texttt{ab}}\texttt{bbbba} \in L$, $\underline{\texttt{bbb}}\texttt{aaabbbaaa} \in L$, $\underline{\texttt{aba}}\texttt{bbbbbbbbb} \in L$, but $\underline{\texttt{a}}\texttt{bbb} \notin L$, $\underline{\texttt{aa}}\texttt{bbbbaa} \notin L$, $\varepsilon \notin L$, $\texttt{bbb} \notin$ L, and $\underline{\texttt{aaa}}\texttt{bbbbbbbbb} \notin L$. Here, the first quarter of the characters in each string has been underlined.

Write a context-free grammar for L.

(ii) Pushdown Automata (10 Points)

*(**Note:** We didn't cover pushdown automata this quarter, so I've removed this question from the practice final.)*

Problem Four: R, RE, and co-RE Languages (60 Points Total)

(i) Properties of Reductions (15 Points)

A language L is called *nontrivial* iff $L \neq \emptyset$ and $L \neq \Sigma^*$.

Prove or disprove: All nontrivial **RE** languages are mapping reducible to one another.
Prove or disprove: All nontrivial co-**RE** languages are mapping reducible to one another.

(ii) RE Languages (20 Points)

Consider the following language, which consists of all Turing machines that reject at least one string:

$$L_{\text{R1}} = \{\ \langle M \rangle \mid M \text{ rejects at least one string }\}$$

Prove that $L_{\text{R1}} \in$ **RE**.

(iii) Unsolvable Problems **(25 Points)**

Prove that $L_{R1} \notin$ co-**RE**. As a reminder, L_{R1} is defined as follows:

$$L_{R1} = \{ \langle M \rangle \mid M \text{ rejects at least one string } \}$$

Problem Five: P and NP Languages (40 Points Total)

(i) Mutual Reducibility **(15 Points)**

Prove that for any language L, L is **NP**-complete iff $L \leq_P$ 3SAT and 3SAT $\leq_P L$.

(ii) Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ **(25 Points)**

Suppose that V is a polynomial-time verifier for an **NP**-complete language L. That is,

$$w \in L \quad \text{iff} \quad \exists x \in \Sigma^*.\ V \text{ accepts } \langle w, x \rangle$$

and

$$V \text{ runs in time polynomial in } |w|$$

Now, suppose that V is a "superverifier" with the property that for any string $w \in L$, V accepts $\langle w, x \rangle$ for almost all choices of x. Specifically, for any $w \in L$, there are at most five strings x for which V rejects $\langle w, x \rangle$.

Under these assumptions, prove that **P** = **NP**.